Complex Modal Analysis of Random Vibrations

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Introduction

A COMPLEX modal analysis of random vibrations is developed in this Note. In contrast to real modal analysis, which is valid only for the classical damping case, complex analysis is valid for both classical and nonclassical damping cases. It is a time domain analysis most suitable for finding the covariances and correlation functions of the random responses.

Analysis

Consider a general n degree-of-freedom linear system, with the following governing differential equation:

$$m\ddot{x}+c\dot{x}+kx=w(t) \tag{1}$$

where m, c, and k are assumed to be $n\times n$ real symmetrical positive-definite matrices and $w(t)$ a stationary white noise excitation, with the following properties:

$$E[w(t)]=0, \qquad E[w(t)w^T(t+\tau)]=2\pi D\delta(\tau)$$

where D is the autospectrum matrix of $w(t)$.

The $2n$ eigenvalues p_i and their corresponding eigenvectors u_i can be found by standard methods. When damping is below critical, the p_i and u_i all appear in conjugate pairs. With them we can construct an $n\times 2n$ complex modal matrix,

$$u=[u_1 \ldots u_{2n}]$$

and a $2n\times 2n$ complex modal matrix,

$$U=[U_1 \ldots U_{2n}]=[Pu^T \;\; u^T]^T$$

where P is an eigenvalue matrix, i.e.,

$$P=\operatorname{diag}[p_i]$$

By introducing state variables x and $\dot{x}$, Eq. (1) can be written as[1]

$$\tilde{m}\dot{y}+\tilde{k}y=\tilde{w}(t) \tag{2}$$

where

$$\tilde{m}=\begin{bmatrix} 0 & m \\ m & c \end{bmatrix}, \quad \tilde{k}=\begin{bmatrix} -m & 0 \\ 0 & k \end{bmatrix}$$

$$y=[\dot{x}^T \;\; x^T]^T, \quad \tilde{w}(t)=[0 \;\; w^T(t)]^T$$

Actually, U_i are the eigenvectors of Eq. (2). It is easy to verify that

$$U^T\tilde{m}U=\operatorname{diag}[m_i]=M$$
$$U^T\tilde{k}U=\operatorname{diag}[k_i]=K \tag{3}$$

with

$$k_i=-p_im_i$$

Hence, by using the complex modal transform,

$$y=Uz$$

Eq. (2) can be reduced to

$$\dot{z}-Pz=F(t) \tag{4}$$

where

$$F(t)=M^{-1}U^T\tilde{w}(t)=[F_i]$$

with

$$E[F(t)]=0$$
$$E[F(t)F^T(t+\tau)]=2\pi M^{-1}u^TD\overline{uM^{-T}}\delta(t)$$

For brevity, let

$$M^{-1}u^TD\overline{uM^{-T}}=G=[g_{is}] \tag{5}$$

Equation (4) may be written in scalar form,

$$\dot{z}_i-p_iz_i=F_i, \quad i=1,2,\ldots,2n$$

Now the system itself is uncoupled, for which the impulse responses are

$$h_i(t)=e^{p_it}, \quad t>0, \quad i=1,2,\ldots,2n$$

The stationary solution of Eq. (4) may be formulated as

$$z(t)=\int_0^\infty h(u)F(t-u)\mathrm{d}u$$

where

$$h(u)=\operatorname{diag}[h_i(u)]$$

Since z is a complex random vector, the correlation function matrix of z may be written as

$$R_z(\tau)=E[z(t)\bar{z}^T(t+\tau)]$$
$$=2\pi\int_0^\infty\int_0^\infty h(u)G\delta(\tau-v+u)\bar{h}^T(v)\mathrm{d}u\mathrm{d}v$$
$$=2\pi\int_0^\infty h(u)G\bar{h}^T(u+\tau)\mathrm{d}u$$

Its element $R_{is}^z(\tau)$ may be obtained as follows:

$$R_{is}^z(\tau)=E[z_i(t)\bar{z}_s(t+\tau)]$$
$$=2\pi\int_0^\infty h_i(u)g_{is}\bar{h}_s(u+\tau)\mathrm{d}u$$
$$=2\pi G_{is}e^{\bar{p}_s\tau}\int_0^\infty e^{(p_i+\bar{p}_s)u}\mathrm{d}u$$
$$=-2\pi e^{\bar{p}_s\tau}g_{is}/(p_i+\bar{p}_s), \quad \tau>0 \tag{6}$$

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with the property

$$R_{is}^{z}(-\tau)=\bar{R}_{si}^{z}(\tau)$$

The covariances may be obtained by letting $\tau=0$ in Eq. (6). We have

$$R_{is}^{z}(0)=E[z_i\bar{z}_s]=-2\pi g_{is}/(p_i+\bar{p}_s)$$

and

$$R_z(0)=E[z\bar{z}^T]=[R_{is}^{z}(0)]$$

Finally, $R_z(\tau)$ may be expressed as

$$R_z(\tau)=R_z(0)\mathrm{diag}[e^{\bar{p}_i\tau}]$$

Returning to Eq. (1) and noting that

$$x=uz$$

we have

$$R_x(\tau)=uR_z(\tau)\bar{u}^T \tag{7}$$

Examples

1) Consider a particular two degree-of-freedom classically damped system by assuming in Eq. (1) that

$$c=\begin{bmatrix} 9 & -1 \\ -1 & 1.5 \end{bmatrix}, \quad k=100c, \quad m=I, \quad 2\pi D=I$$

The four eigenvalues are

$$p_1=\bar{p}_3=A_1+jB_1=-0.68448+j11.6802$$

$$p_2=\bar{p}_4=A_2+jB_2=-4.56552+j29.8078$$

and the four eigenvectors are

$$u_1=u_3=[a_{11}\ \ a_{21}]^T=[0.12993 \quad 0.99152]^T$$

$$u_2=u_4=[a_{12}\ \ a_{22}]^T=[0.99152 \quad -0.12993]^T$$

From Eq. (3), we have

$$M=U^T\bar{m}U=2j\begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix}, \quad B=\mathrm{diag}[B_i]$$

and by Eq. (5),

$$2\pi G=M^{-1}u^T\overline{uM}^{-T}$$

$$=(1/4)\begin{bmatrix} g & -g \\ -g & g \end{bmatrix}, \quad q=\begin{bmatrix} B_1^{-2} & 0 \\ 0 & B_2^{-2} \end{bmatrix}$$

Then by Eq. (6), we get $R_z(\tau)$. And by Eq. (7), we finally obtain

$$R_x(\tau)=\begin{bmatrix} \Sigma a_{1i}^2 r_i(\tau) & \Sigma a_{1i}a_{2i}r_i(\tau) \\ \Sigma a_{1i}a_{2i}r_i(\tau) & \Sigma a_{2i}^2 r_i(\tau) \end{bmatrix}$$

where

$$r_i=\frac{1}{B_i^2}\left[\left(\frac{1}{\bar{p}_i}-\frac{1}{A_i}\right)e^{\bar{p}_i\tau}+\left(\frac{1}{p_i}-\frac{1}{A_i}\right)e^{p_i\tau}\right], \quad i=1,2$$

The result is exactly the same as that obtained by real modal analysis.[2]

Table 1 Numerical results for $R_x[\tau]$

τ	$R_{11}(\tau)$	$R_{12}(\tau)$	$R_{21}(\tau)$	$R_{22}(\tau)$
0.00	0.28437	0.50546	0.50546	1.22290
0.04	0.17644	0.33459	0.35022	0.72992
0.08	−0.05920	−0.07176	−0.04372	−0.26742
0.12	−0.23959	−0.45240	−0.42903	−0.94252
0.16	−0.23973	−0.51280	−0.51692	−0.90517
0.20	−0.07417	−0.17974	−0.21249	−0.31091
0.24	0.13485	0.29221	0.25878	0.44576
0.28	0.24660	0.52917	0.52437	0.93281
0.32	0.18646	0.36389	0.38840	0.80475
0.36	−0.00660	−0.04933	−0.01840	0.06692
0.40	−0.19724	−0.40011	−0.38480	−0.76872
0.44	−0.24442	−0.46046	−0.46702	−1.02624
0.48	−0.11119	−0.20794	−0.23013	−0.48621
0.52	0.10147	0.18533	0.16019	0.39776
0.56	0.23422	0.45299	0.44092	0.93181
0.60	0.19395	0.39877	0.41057	0.76967

2) Consider a two degree-of-freedom nonclassically damped linear system by assuming in Eq. (1) that

$$m=\begin{bmatrix} 100 & 0 \\ 0 & 15 \end{bmatrix}, \quad c=\begin{bmatrix} 75 & -50 \\ -50 & 50 \end{bmatrix}$$

$$k=\begin{bmatrix} 76915 & -14415 \\ -14415 & 14415 \end{bmatrix}, \quad 2\pi D=I$$

The four eigenvalues are

$$p_1=\bar{p}_3=-0.2337+j21.938$$

$$p_2=\bar{p}_4=-1.8079+j35.281$$

and the four eigenvectors are

$$u_1=\bar{u}_3=[1 \quad 1.9901-j0.1085]^T$$

$$u_2=\bar{u}_4=[1 \quad -3.3432-j0.2942]^T$$

By following the procedure illustrated in the first example, the numerical results for $R_x(\tau)$ are shown in Table 1.

Conclusions

1) A complex modal method for stationary random response problems due to arbitrarily correlated white noise excitation is presented. The method is applicable to any kind of linear damping. All of the calculations can be programmed for a computer.

2) By using both left and right modal matrices, the method can also be applied to nonsymmetrical systems.[2]

3) By enlarging the original system to include some filtering process, the method is also applicable for filtered white noise excitations.[2]

4) Based on these results, the authors have successfully generalized Caughey's normal mode approach to nonlinear random vibration problems.[3,4]

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Finite Element Analysis of Elastoplastic Contact Problems with Friction

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Introduction

THE aim of this Note is to extend the authors' previous work[1] analyzing elastodynamic sliding contact problems with friction to deal with static small deformation elastoplastic problems. In this work, the deformed contact areas are obtained by the constraint conditions developed by a quadratic mathematic programming technique.[2] Lagrangian multipliers are introduced to evaluate the contact pressures due to friction and determine the adhesion or release of contact surface. The plastic behaviors in the contact components are accurately modeled. The influence of material properties and friction effects on the plastic zones and deformations is discussed. This is of practical importance in designing the optimum metal-forming process, especially when using a deformable die, a case which is seldom solved in literature.[3]

Finite Element Formulation

Following the procedure developed by the authors,[1] the governing equations for elastoplastic contact problems with friction can be written as

$$\frac{\partial \Pi_M}{\partial\{{}^{\Delta U_i}_{T_S}\}}=0 \tag{1}$$

$$\sum_{\ell=1}^{L}\int_{S_{C\ell}}\lambda[\mathfrak{F}(X_{Di}^{(N+1)})-\mathfrak{F}(X_{Wi}^{(N+1)})]\,\mathrm{d}S=0 \tag{2}$$

$$\sum_{\ell=1}^{L}\int_{S_{C\ell}}[\mathfrak{F}(X_{Di}^{(N+1)})-\mathfrak{F}(X_{Wi}^{(N+1)})]\,\mathrm{d}S\geq 0 \tag{3}$$

and

$$\lambda\geq 0 \tag{4}$$

where ΔU_i are the incremental displacement components from the Nth load step to the $(N+1)$th step; T_S the Lagrangian multiplier, which can be shown to be the tangential contact pressure due to friction; λ another Lagrangian multiplier used to relax the condition of non-interpenetration on element contact surface $S_{C\ell}$; L the total number of element contact surfaces; and $\mathfrak{F}$ the contact surface function that is taken to establish the non-interpenetration conditions between components D and W. The variables $X_{Di}^{(N+1)}$, $X_{Wi}^{(N+1)}$ represent the Cartesian coordinates of the material points in contact components D and W. The functional Π_M can be expressed as

$$\Pi_M(\Delta U_i,T_S,\lambda)=\sum_{n=1}^{N_D}\left[\int_{\Omega Dn}\left(\frac{1}{2}\Delta\sigma_{ij}\Delta\epsilon_{ij}\right)\mathrm{d}V-\int_{S_{\sigma Dn}}(\Delta\bar{T}_i\Delta U_i)\,\mathrm{d}S+\epsilon_{Dn}^{(N)}\right]$$
$$+\sum_{n=1}^{N_W}\left[\int_{\Omega Wn}\left(\frac{1}{2}\Delta\sigma_{ij}\Delta\epsilon_{ij}\right)\mathrm{d}V-\int_{S_{\sigma Wn}}(\Delta\bar{T}_i\Delta U_i)\,\mathrm{d}S+\epsilon_{Wn}^{(N)}\right]$$
$$-\sum_{\ell=1}^{L}\left[\int_{S_{C\ell}}T_S(\Delta U_{DS}-\Delta U_{WS})\,\mathrm{d}S\right]$$
$$-\sum_{\ell=1}^{L}\left[\int_{S_{C\ell}}\lambda(\mathfrak{F}(X_{Di}^{(N+1)})-\mathfrak{F}(X_{Wi}^{(N+1)}))\,\mathrm{d}S\right]$$

where

$$\epsilon_{Dn}^{(N)}=\int_{\Omega Dn}\sigma_{ij}^{(N)}\Delta\epsilon_{ij}\,\mathrm{d}V-\int_{S_{\sigma Dn}}\bar{T}_i^{(N)}\Delta U_i\,\mathrm{d}S$$

$$\epsilon_{Wn}^{(N)}=\int_{\Omega Wn}\sigma_{ij}^{(N)}\Delta\epsilon_{ij}\,\mathrm{d}V-\int_{S_{\sigma Wn}}\bar{T}_i^{(N)}\Delta U_i\,\mathrm{d}S$$

ΔU_i, T_S, and λ are treated as independent variables in Π_M. $(N_D, \Omega_{Dn}, S_{\sigma Dn})$ and $(N_W, \Omega_{Wn}, S_{\sigma Wn})$ represent the total number of elements, element domains, and element traction surfaces of components D and W, respectively. $\Delta\sigma_{ij}$ and $\Delta\epsilon_{ij}$ are the incremental stress and strain tensors from the Nth load step to the $(N+1)$th step, and $\Delta\bar{T}_i$ are the incremental prescribed tractions from the Nth load step to the $(N+1)$th step. $\epsilon_{Dn}^{(N)}$ and $\epsilon_{Wn}^{(N)}$ are the correction terms to do the equilibrium check of components D and W at the Nth load step incurred by nonlinear plastic behaviors. $\sigma_{ij}^{(N)}$ is the stress tensor and $\bar{T}_i^{(N)}$ are the total prescribed tractions at the Nth load step. The tangential components of incremental displacements ΔU_i on contact surfaces of components D and W are denoted as ΔU_{DS} and ΔU_{WS}, respectively.

Closely following the authors' earlier work,[1] some tedious manipulations resulted in the final simultaneous algebraic equations and constraints

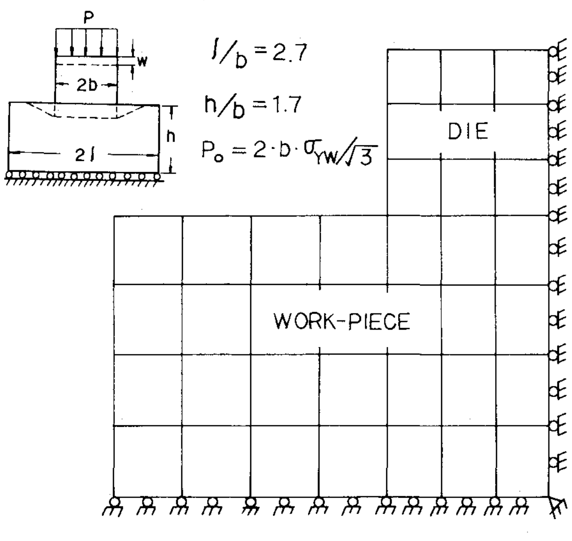

Fig. 1 ADINA's punch problem.

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